MATB41 Week 6 Notes

1. Difference Between ∂ and d.

∂ is used when there are other variables in the final solution.

E.g.

Let $f = xy$

$$\frac{\partial f}{\partial x} = y \leftarrow \text{Another variable}$$

Let $g = x$

$$\frac{dg}{dx} = 1$$

Note: You can have more than 1 variable in the function and still use d.

2. Chain Rule

Let $w = f(x, y, z, t)$
Let $x = x(t)$
Let $y = y(t)$
Let $z = z(t)$

$$\frac{dw}{dt} = \left(\frac{\partial w}{\partial x} \cdot \frac{dx}{dt}\right) + \left(\frac{\partial w}{\partial y} \cdot \frac{dy}{dt}\right) + \left(\frac{\partial w}{\partial z} \cdot \frac{dz}{dt}\right) + \left(\frac{\partial w}{\partial t}\right)$$

E.g. Let $z = u^2 \sin(v) + 2t$
let $u = \sin(t)$
let $v = t^3$

Find $\frac{dz}{dt}$

Solution:

$$\frac{dz}{dt} = \frac{\partial z}{\partial u} \cdot \frac{du}{dt} + \frac{\partial z}{\partial v} \cdot \frac{dv}{dt} + \frac{\partial z}{\partial t}$$

$$= (2u\sin v)(\cos t) + (u^2 \cos v)(3t^2) + 2$$

$$= (2)(\sin t)(\sin t^3)(\cos t) + (\sin^2 t)(\cos t^3)(3t^2) + 2$$

See how we started with 3 variables but still used d because the final answer only has 1 variable.

3. Jacobian Matrix:

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

$\leftarrow$ Each row has the same function.

$\uparrow$
Each col has the same variable.

4. Tangent/Velocity Vectors:

Let c be a path defined by $c(t) = (x(t), y(t), z(t))$ and let c be differentiable.

The tangent vector of c at t is defined by

$$c'(t) = \lim_{h \to 0} \frac{c(t+h) - c(t)}{h}, \quad \text{where } h = \Delta t$$

$$= (x'(t), y'(t), z'(t)).$$

Proof:

$$c'(t) = \lim_{h \to 0} \frac{c(t+h) - c(t)}{h}$$

$$= \lim_{h \to 0} \frac{(x(t+h), y(t+h), z(t+h)) - (x(t), y(t), z(t))}{h}$$

$$= \lim_{h \to 0} \frac{(x(t+h) - x(t), y(t+h) - y(t), z(t+h) - z(t))}{h}$$

$$= \left(\lim_{h \to 0} \frac{(x(t+h) - x(t))}{h}, \lim_{h \to 0} \frac{(y(t+h) - y(t)}{h}, \lim_{h \to 0} \frac{(z(t+h) - z(t))}{h} \right)$$

$$= (x'(t), y'(t), z'(t)).$$

E.g. Compute the tangent vector to the path of $c(t) = (t, t^2, e^t)$ at $t=0$.

Soln: $c'(t) = (1, 2t, e^t)$
Plugging in t, we get $(1,0,1)$, which is the tangent vector.

5. Tangent Line:

The tangent line to c at point $a = (x(t_0), y(t_0), z(t_0))$ is defined to be the line through "a" with a direction of $c'(t_0)$.

Formula:

$$\underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} x(t_0) \\ y(t_0) \\ z(t_0) \end{pmatrix}}_{a} + \underbrace{\begin{pmatrix} x'(t_0) \\ y'(t_0) \\ z'(t_0) \end{pmatrix}}_{c'(t_0)} (t - t_0)$$

OR

$$\vec{x} = a + c'(t_0)(t - t_0)$$

E.g. A path in R^3 goes through the point $(3, 6, 5)$ at $t=0$ with a tangent vector $i - j$. Find the eqn of the tangent line.

Soln:

$t = 0$ means that $t_0 = 0$.

$$\vec{x} = (3, 6, 5) + t(1, -1, 0)$$
$$= (3+t, 6-t, 5)$$

OR

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 5 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

E.g. Find the velocity vector of the path $c(t) = (\cos t, \sin t, t)$. Then, find the tangent line of the curve at $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{\pi}{4})$.

Soln:

$a = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{\pi}{4})$

$c'(t) = (-\sin t, \cos t, 1)$

$x(t_0) = \frac{1}{\sqrt{2}} \longrightarrow \cos(t_0) = \frac{1}{\sqrt{2}}$

$y(t_0) = \frac{1}{\sqrt{2}} \longrightarrow \sin(t_0) = \frac{1}{\sqrt{2}}$

$z(t_0) = \frac{\pi}{4} \longrightarrow t_0 = \frac{\pi}{4}$

Note: Check that the value of t_0 fits with the other eqns.

In our case,
$\sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$
$\cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

$$c'(t_0) = c'(\tfrac{\pi}{4}) = (-\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 1)$$

$\therefore$ The tangent line is

$$x = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}(t - \frac{\pi}{4})$$

$$y = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}(t - \frac{\pi}{4})$$

$$z = \frac{\pi}{4} + (t - \frac{\pi}{4})$$

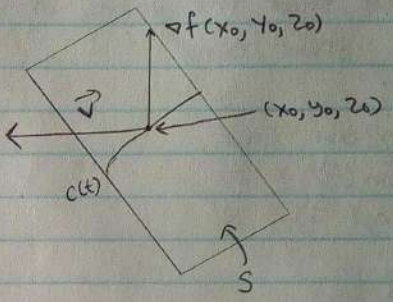
6. The gradient is normal to level surfaces:

Let $f: \mathbb{R}^3 \to \mathbb{R}$ have cont partial derivatives and let (x_0, y_0, z_0) lie on the level surface S defined by $f(x,y,z) = k$, k is a constant. Then, $\nabla f(x_0, y_0, z_0)$ is normal (orthogonal) to S.

$$S = \{(x, y, z) \mid f(x,y,z) = k\}$$

Another way to think about this is:

If $\vec{v}$ is a tangent vector at $t=0$ of a path $c(t)$ in S with $c(0) = (x_0, y_0, z_0)$, then $\nabla f(x_0, y_0, z_0) \cdot \vec{v} = 0$.



Proof:
Let $c(t) = (x(t), y(t), z(t))$ be any differentible path that passes x_0 at $t = t_0$ on S. $v = c'(t_0)$

$\therefore f(x(t_0), y(t_0), z(t_0)) = k$, $k \in \mathbb{R}$

$$\frac{df}{dt_0} = \left(\frac{\partial f}{\partial x}\right)\left(\frac{dx}{dt_0}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{dy}{dt_0}\right) + \left(\frac{\partial f}{\partial z}\right)\left(\frac{dz}{dt_0}\right)$$

$$\frac{df}{dt_0} = \frac{dk}{dt_0}$$

$= 0$ (k is a constant and the derivative of constants = 0.)

$$\frac{df}{dt_0} = \left(\frac{\partial f}{\partial x}\right)\left(\frac{dx}{dt_0}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{dy}{dt_0}\right) + \left(\frac{\partial f}{\partial z}\right)\left(\frac{dz}{dt_0}\right)$$

$$= \underbrace{\left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]}_{\nabla f(x,y,z)} \cdot \underbrace{\left[\frac{dx}{dt_0}, \frac{dy}{dt_0}, \frac{dz}{dt_0}\right]}_{\vec{v}}$$

$\therefore 0 = \nabla f(x,y,z) \cdot \vec{v}$

7. Tangent Planes:

Let S be the surface containing (x,y,z) s.t. $f(x,y,z) = k$, k is a constant. The tangent plane of S at a point (x_0, y_0, z_0) of S is defined by the eqn:

$\nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$
if $\nabla f(x_0, y_0, z_0) \neq 0$.

Proof:

Let $c(t) = (x(t), y(t), z(t))$ be any differentible path that passes through (x_0, y_0, z_0) at $t = t_0$ on S.

$(x - x_0,\ y - y_0,\ z - z_0) = (c'(t_0))(t - t_0)$

This is the tangent line of $c(t)$ at $t = t_0$.

$\nabla f(x(t_0), y(t_0), z(t_0)) \cdot (c'(t_0))(t - t_0)$

$= 0(t - t_0)$

$= 0$

E.g. Find the eqn of the plane tangent to the surface defined by $3xy + z^2$ at $(1,1,1)$.

Solution:

$f(x,y,z) = 3xy + z^2$

$\nabla f(x,y,z) = [3y,\ 3x,\ 2z]$

At the point $(1,1,1)$, $\nabla f = [3,3,2]$.

$(3,3,2) \cdot (x-1,\ y-1,\ z-1) = 0$

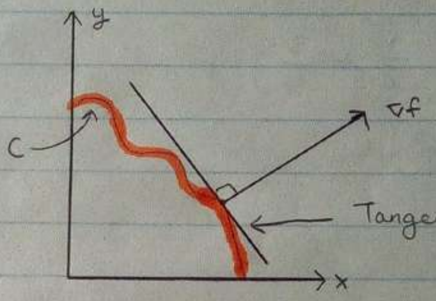
$3(x-1) + 3(y-1) + 2(z-1) = 0$

$3x + 3y + 2z = 8$

What we've just learned is also applicable for a function $f: R^2 \to R$. However, this time, we will be using level curves.

Let $f: R^2 \to R$. Let C be a level curve containing (x, y) s.t. $f(x,y) = k$, k is a constant. Then, $\nabla f(x_0, y_0)$ is perpendicular to C for any point (x_0, y_0) on C.

$$\nabla f(x_0, y_0) \cdot (x - x_0, y - y_0) = 0 \quad \text{if} \quad \nabla f(x_0, y_0) \neq 0$$



E.g. Find the points on the surface defined by $x^2 + 2y^2 + 3z^2 = 1$ where the tangent plane is parallel to the plane defined by $3x - y + 3z = 1$.

Soln:

$f(x,y,z) = x^2 + 2y^2 + 3z^2$

$\nabla f = [2x, 4y, 6z]$

$[2x, 4y, 6z] \parallel [3, -1, 3]$

This means that there exists a k such that

$2x = 3k \rightarrow x = \frac{3}{2}k$

$4y = -k \rightarrow y = -\frac{1}{4}k$

$6z = 3k \rightarrow z = \frac{1}{2}k$

Furthermore, we know that $x^2 + 2y^2 + 3z^2 = 1$.

$$\left(\frac{3}{2}k\right)^2 + 2\left(-\frac{1}{4}k\right)^2 + 3\left(\frac{1}{2}k\right)^2 = 1$$

$$\frac{9k^2}{4} + \frac{2k^2}{16} + \frac{3k^2}{4} = 1$$

$$\frac{9k^2}{4} + \frac{k^2}{8} + \frac{3k^2}{4} = 1$$

$$k^2\left(\frac{9}{4} + \frac{1}{8} + \frac{3}{4}\right) = 1$$

$$k^2\left(\frac{18+1+6}{8}\right) = 1$$

$$k^2 = \frac{8}{25}$$

$$k = \pm\frac{2\sqrt{2}}{5}$$

Solving for x, y, and z, we get

1. $x = \frac{3\sqrt{2}}{5}$, $y = -\frac{\sqrt{2}}{10}$, $z = \frac{\sqrt{2}}{5}$

2. $x = -\frac{3\sqrt{2}}{5}$, $y = \frac{\sqrt{2}}{10}$, $z = -\frac{\sqrt{2}}{5}$

$\therefore$ The two points are

1. $\left(\frac{3\sqrt{2}}{5}, \frac{-\sqrt{2}}{10}, \frac{\sqrt{2}}{5}\right)$

2. $\left(-\frac{3\sqrt{2}}{5}, \frac{\sqrt{2}}{10}, -\frac{\sqrt{2}}{5}\right)$

8. Linear Approximation:

Let $f: R^2 \to R$ be differentible at (x_0, y_0).
The linear approx of f at (x_0, y_0) is defined by

$$L(x,y) = f(x_0, y_0) + \left[\frac{\partial f}{\partial x}(x_0, y_0)\right](x - x_0) + \left[\frac{\partial f}{\partial y}(x_0, y_0)\right](y - y_0)$$

E.g. Find the linear approx to the function $f(x,y) = \sin(xy)$ at $(1, \frac{\pi}{3})$.

Soln:

$$f(1, \tfrac{\pi}{3}) = \sin(\tfrac{\pi}{3}) = \frac{\sqrt{3}}{2}$$

$$L(x) = f(1, \tfrac{\pi}{3}) + \frac{\partial f}{\partial x}(1, \tfrac{\pi}{3})(x-1) + \frac{\partial f}{\partial y}(1, \tfrac{\pi}{3})(x-1)$$

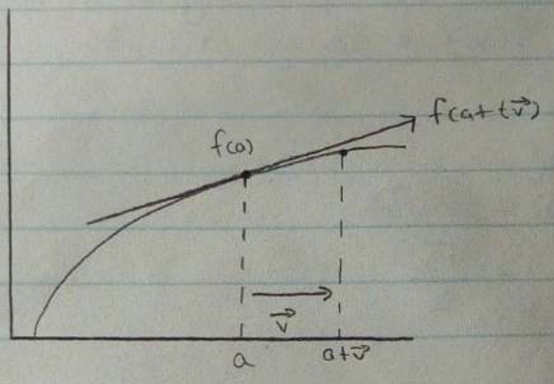
$$= \frac{\sqrt{3}}{2} + \left(\frac{\pi}{3}\right)\left(\sin\left(\frac{\pi}{3}\right)\right)(x-1) + (1)\left(\sin\left(\frac{\pi}{3}\right)\right)(y-1)$$

$$= \frac{\sqrt{3}}{2} + \frac{\pi}{6}(x-1) + \frac{1}{2}(y-1)$$

9. Directional Derivatives with Linear Approx:

Consider a path along direction $\vec{v}$ that passes through point a. The rate of change of f in the direction $\vec{v}$ is given by $D_{\vec{v}}(f(a))$.

$$D_{\vec{v}}(f(a)) = \lim_{t \to 0} \frac{f(a+t\vec{v}) - f(a)}{t\|\vec{v}\|}$$



We can approx $f(a+t\vec{v})$.

$$L(a+t\vec{v}) = f(a) + D_{\vec{v}}(f(a)) \cdot \|\vec{v}\|$$
$$= f(a) + \nabla f \cdot \vec{v}$$

$D_{\vec{v}}(f(a)) = \nabla f \cdot \dfrac{\vec{v}}{\|\vec{v}\|}$, if $\vec{v}$ is NOT a unit vector

$D_{\vec{v}}(f(a)) = \nabla f \cdot \vec{v}$, if $\vec{v}$ is a unit vector.

If f is differentiable then all directional derivatives exist. Let $\vec{v} = [v_1, v_2, v_3]$ be an unit vector. Then, the directional derivative of f at (x,y,z) in the direction $\vec{v}$ is given by

$$D_{\vec{v}}(f(x,y,z)) = \nabla f(x,y,z) \cdot \vec{v}$$
$$= \frac{\partial f}{\partial x}(x,y,z) \cdot v_1 + \frac{\partial f}{\partial y}(x,y,z) \cdot v_2 +$$
$$\frac{\partial f}{\partial z}(x,y,z) \cdot v_3$$

10. Examples From Tutorial:

1. Partial Derivatives:

Let $f(x,y) = \dfrac{\cos(xy^2)}{x}$

$$\frac{\partial f}{\partial x} = \frac{(y^2)(-\sin(xy^2))(x) - \cos(xy^2)}{x^2}$$

$$\frac{\partial f}{\partial y} = \frac{(2xy)(-\sin(xy^2))}{x}$$
$$= -2y(\sin(xy^2))$$

2. Jacobian Matrix:

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & & & \vdots \\ \vdots & & & \\ \frac{\partial f_k}{\partial x_1} & \cdots & \cdots & \frac{\partial f_k}{\partial x_n} \end{bmatrix} \leftarrow \text{Jacobian Matrix}$$

Let $f(x,y) = (x^2 + x\sin(y),\ x^3 e^{xy},\ 3^{xy}$

$f_1 = x^2 + x\sin(y)$
$f_2 = x^3 e^{xy}$
$f_3 = 3^{xy}$

$$Df(a) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} \end{bmatrix}$$

$$\begin{bmatrix} 2\sin(y) & x\cos(y) \\ e^{xy}(3x^2 + yx^3) & x^4 e^{xy} \\ (y)(\ln 3)(3^{xy}) & (x)(\ln 3)(3^{xy}) \end{bmatrix}$$

Let $f: R^n \to R$

$$Df(a) = \left[\frac{\partial f}{\partial x_1}(a), \frac{\partial f}{\partial x_2}(a), \dots \frac{\partial f}{\partial x_n}(a)\right]$$

This is called the gradient of f, denoted by ∇f.

3. Chain Rule:

$$D(f \circ g)(a) = Df(g(a)) \cdot Dg(a)$$

This is the formula for the chain rule. However, there are 2 special cases of the chain rule.

Case 1:

Suppose we have $f(x, y, z)$, $x(t)$, $y(t)$ and $z(t)$.

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt}$$

Case 2:

Suppose we have $z = f(x, y)$, $x = g(s, t)$, $y = h(s, t)$.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

E.g. Let $z = e^{2x} \cdot \sin(3y)$

Let $x = st - t^2$

Let $y = \sqrt{s^2+t^2}$

Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$

Solution:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$= (2)(e^{2x})(\sin(3y))(t) + (3)(e^{2x})(\cos(3y))\left(\frac{2s}{2\sqrt{s^2+t^2}}\right)$$

$$= (2t)(e^{2(st-t^2)})(\sin(3(\sqrt{s^2+t^2})) + (3)(e^{2(st-t^2)})$$
$$(\cos(3(\sqrt{s^2+t^2}))$$
$$\left(\frac{s}{\sqrt{s^2+t^2}}\right)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$= (2)(e^{2x})(\sin(3y))(s-2t) + (3)(e^{2x})(\cos(3y))\left(\frac{2t}{2\sqrt{s^2+t^2}}\right)$$

$$= (2)(e^{2(st-t^2)})(\sin(3\sqrt{s^2+t^2}))(s-2t) + (3)(e^{2(st-t^2)})$$
$$(\cos(3\sqrt{s^2+t^2}))$$
$$\left(\frac{t}{\sqrt{s^2+t^2}}\right)$$